A SURVEY OF CERTAIN EXTREMAL PROBLEMS FOR NON-VANISHING ANALYTIC FUNCTIONS

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Abstract. This paper surveys a large class of nonlinear extremal problems in Hardy and Bergman spaces. We discuss the general approach to such problems in Hardy spaces developed by S. Ya. Khavinson in the 1960s, but not well known in the West. We also discuss the major difficulties distinguishing the Bergman space setting and formulate some open problems.

1. Introduction

Solving extremal problems has been one of the major stimuli for progress in complex analysis, starting with the Schwarz lemma, on to the celebrated problems of Carathéodory-Fejér, Kakeya, Landau, etc.., (see the historical notes in [14], pp. 51-54 and pp. 110-112), and finally to general linear problems in Hardy spaces. Since the introduction of methods of functional analysis (the Hahn-Banach theorem) in the study of linear extremal problems in analytic function spaces by S. Ya. Khavinson in 1949 ([13]) and, independently, by Rogosinski and Shapiro in 1953 ([25]), the theory of extremal problems in Hardy spaces has achieved a significant level of elegance and clarity (cf. [7], Ch. 8).

Recently, substantial progress has occurred in the twin theory of linear extremal problems in Bergman spaces (see [12, 8, 9] and the references cited there). In this brief survey we are mostly concerned with the problems that are not covered by the elegant umbrella of clean and simple methods of functional analysis, namely, non-linear extremal problems. More precisely, we consider here some well-known basic extremal problems such as finding the maximum value of a simple linear functional, but posed for *non-vanishing* functions in either Hardy or Bergman spaces.

The latter set of functions is obviously non-convex, and accordingly, new methods are required to solve problems in this new setting. A celebrated example of a problem that is still far from being solved is the Krzyż conjecture for bounded non-vanishing analytic functions. Namely, if we consider the family $\mathcal{F}$ consisting of all non-vanishing, bounded analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ such that

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$|f(z)| \leq 1$ for $|z| < 1$, the Krzyż conjecture states that, for $m \geq 1$,

$$\max\{|a_m| : f \in \mathcal{F}\} = \frac{2}{e}.$$

This conjecture has been proven only for $1 \leq m \leq 5$ (see [10, 11, 16, 18, 17, 20, 21, 22, 23, 24, 26, 27, 31, 32, 30, 33, 35]). At the same time, if we considered the *linear* analogue of this question by removing the condition that $f(z) \neq 0$ in $\mathbb{D}$, then the problem is trivial and the extremal functions $f^*(z) = e^{i\alpha}z^m$ give the value 1 for the maximum. S. Ya. Khavinson developed, in the early '60s, a general approach to problems for non-vanishing functions in Hardy spaces that allowed him, if not to solve the problem explicitly, to at least obtain the particular form of extremal functions. Yet, he did not publish it until the 1970s. Moreover, the latter work was not translated into English until 1986 (see [14]). Under his guidance, his former student, V. Terpigoreva, quickly extended his results to more general Orlicz-Hardy spaces in the paper [37] following her thesis [36]. She published a complete version with proofs in 1970 (see [38]). This perhaps partly explains why Khavinson postponed publication of his less general results until their inclusion in his monograph ([14]) that unfortunately was never published in Russian in book form.

Some of S. Ya. Khavinson's results (but not the general method) were rediscovered in the 70s and 80s by western authors (see [11, 32]). Yet, the attack on extremal problems for non-vanishing functions in Bergman spaces has only just begun (see [1, 2, 3, 4, 5]), and still, the simplest problems remain unsolved.

The layout of this survey is as follows. In Section 2, we outline S. Ya. Khavinson's theory for Hardy spaces. In Section 3, we illustrate the general theory by discussing some particular examples in Hardy spaces. Section 4 contains the discussion of the Bergman space case. There we focus on the simplest problems that are still unresolved; in particular, we explain in detail where S. Ya. Khavinson's arguments that work so smoothly for Hardy spaces run into a wall in the Bergman space context. We finish with several observations and conjectures for the Bergman spaces problem that we hope will attract more researchers to this field.

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2. General Theory for Hardy Spaces

Let us begin by discussing the general theory of coefficient type extremal problems for non-vanishing functions in Hardy spaces. This discussion is based on the work of S. Ya. Khavinson in [14]. (The results there were originally obtained in the mid 60s, yet the original version of [14] was only published in Russian in 1981.)

We shall be looking at a general extremal problem of the following type: given

(242) In other words, we are interested in finding, given $\gamma_0, \gamma_1, \dots, \gamma_m \in \mathbb{C}$ fixed,

$$\lambda_p^* = \sup \left\{ \mathrm{Re} \sum_{k=0}^{m} \gamma_k a_k : q(z) = \sum_{k=0}^{\infty} a_k z^k \in Q_p^* \right\}. \tag{2.2}$$

Now examine the structure of such functions q a little more closely.

It is well-known (see [7]) that every function $f \in H_0^p$ has non-tangential limits (almost everywhere on the unit circle $\mathbb{T}$) $f(e^{it}) \in L^p([0, 2$ f

A simple calculation shows that

$$\operatorname{Re}\left(\sum_{k=0}^{m}\gamma_k a_k\right) = \frac{1}{2\pi p}\int_0^{2\pi}\alpha(t)S(t)\,dt; \tag{2.9}$$

where

$$\alpha(t) := \operatorname{Re}\left(\gamma_0 + 2\sum_{k=1}^{m}\gamma_k e^{-ikt}\right)$$

and S "represents" q via (2.7), that is, $S(t) = p\operatorname{Re} q(e^{it}).$

Now if $\alpha(t)$ is continuous on the interval $[0, 2\pi]$, it is not hard to see that the supremum

$$\sup_{S\in\sigma}\int_0^{2\pi}S(t)\alpha(t)\,dt \tag{2.10}$$

is finite if and only if $\alpha(t) \geq 0$ on $[0, 2\pi].$ Indeed, if $\alpha(t) \geq 0$, then

$$\int_0^{2\pi}S(t)\alpha(t)\,dt \leq \int_0^{2\pi}e^{S(t)}\alpha(t)\,dt < \infty;$$

since α is continuous on $[0, 2\pi]$ and S satisfies the normalization (2.5). On the other hand, suppose there were an interval I and $\epsilon > 0$ such that $\alpha(t) < -\epsilon$ for $t \in I.$ Then we could construct a sequence of functions S_N equal to $-N$ on that interval I and 0 elsewhere. The measures $S_N(t)\,dt$ certainly lie in the class σ, and the integrals

$$\int_0^{2\pi}S_N(t)\alpha(t)\,dt \to \infty.$$

Similarly,

$$\sup_{\nu\in\Sigma}\int_0^{2\pi}\alpha(t)d\nu(t) < \infty$$

Notice that it is enough to consider $S \in \sigma$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{S(t)}\,dt = 1.$$

Now, the following simple inequality can be checked directly, for any $u, v > 0$:

$$u \ln u - u \geq u \ln v - v. \qquad (2.13)$$

Moreover, if $u \neq v$; then the inequality is strict. Applying (2.13) to $u = \alpha v_u -$

functions $f \in H^p$ with prescribed initial coefficients $c_0, c_1, \ldots, c_m$. The same type of argument also applies to interpolation problems where the origin is replaced by arbitrary points in the unit disk.

3. Some specific problems in Hardy Spaces

The authors note that for $p = 1$, the functions

$$f_m(z) = \frac{(1 + z^m)^2}{2}$$

are extremal for Problem (3.1) and, for all $m \geq 1$,

$$\sup_{f \in H^1_0} \operatorname{Re} \left\{ \frac{f^{(m)}(0)}{m!} \right\} = 1. \tag{3.4}$$

They also remark that uniqueness of extremals fails badly here: any function

$$f(z) = C \prod_{j=1}^{m} (z - \alpha_j)(1 - \bar{\alpha}_j z),$$

where $|\alpha_j| = 1$ and C is chosen so that $\|f\|_{H^1} = 1$ is also extremal. Note that all of these extremals are of the general form (2.16) from Section 2. Conjecture 3.3 for non-vanishing Hardy space functions was shown to be true for $m = 1$ in [6] and for $m = 2$ in [32].

While studying explicit solutions to *linear* extremal problems in H^p, the authors in [5] considered the related problem of finding, for $p \geq 1$, $m \geq 1$, and $0 < c < 1$ fixed,

$$\max_{f \in H^p_0} \left\{ \operatorname{Re} f^{(m)}(0)/m! : \ f(0) = c \right\}. \tag{3.5}$$

They were only able to solve this problem explicitly for $m = 1$. The extremals depend on the value of c : if $0 < c < 2^{-1/p}$, then the extremal function has a singular part and is equal to

$$f^*(z) = 2^{-\frac{1}{p}}(1 + z)^{\frac{2}{p}} \exp\left(-\gamma_0 \frac{-1 + z}{-1 - z} \right),$$

where $\gamma_0 = -\log(2^{\frac{1}{p}} c)$, while if $2^{-1/p} \leq c < 1$, then the extremal has no singular part and is equal to

$$f^*(z) = \left(c^{\frac{p}{2}} + z\sqrt{1 - c^p} \right)^{\frac{2}{p}}.$$

By varying c and finding the corresponding maximum, the authors gave another proof of Conjecture (3.3) for $m = 1$. Problem (3.5) is equivalent to an interpolation problem for non-vanishing functions, of finding, for $c_0, \ldots, c_m$ fixed,

$$\inf\{\|f\|_p : \ f(0) = c_0, \ldots, \frac{f^{(m)}(0)}{m!} = c_m; \ f \in H^p; \ f \text{ non-vanishing } \}.$$

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occurs exactly at the zero of the outer part of f^*; making the extremal function continuous in the closed unit disk. This example will be discussed in more detail in the Bergman space setting in the following section.

using a more delicate variation stemming from the seminal work in [2, 3, 4] on the so-called minimal area problem, i.e., the problem of finding, for b fixed,

$$\inf\left\{\int_{\mathbb{D}} |F'|^2\,dA : F(0) = 0, F'(0) = 1, F''(0) = b, F \text{ univalent in } \mathbb{D}\right\}; \tag{4.3}$$

it was shown in [1] that the extremal f^* is in fact bounded in $\mathbb{D}$. It was conjectured in [1] that the extremal function f^* has the form (for $p = 2$):

$$f^* \tag{4.4}$$

into C^{m+1}. Following S. Ya. Khavinson's scheme from Section 2, it is straightforward to show that the image $A_r := \Phi(\S_r)$ is a closed, convex, proper subset of C^{m+1} with non-empty interior. If we denote by $\vec{a} = < a_0, \ldots, a_m >$ the vector of coordinate data in (4.2), then the value of the infimum in (4.2) equals

$$r_0 = \inf\{r > 0 :$$

Using the complex form of Green's theorem together with the fact that

$$\left|\exp\left(\mu\frac{1+z}{z-1}\right)\right| = 1 \text{ a.e. on } \mathbb{T},$$

we calculate

$$\begin{aligned}
\int_{\mathbb{D}}|f^*(z)|^2\,dA &= \frac{i}{2\pi}\int_{\mathbb{T}}F^*(z)\overline{f^*(z)}\,d\bar z\\
&= \frac{i}{2\pi}\int_{\mathbb{T}}F^*(z)\overline{f^*(z)}(-i)\bar z\,d\theta\\
&= \frac{1}{2\pi}\int_{\mathbb{T}}\frac{C}{2}(z-1)^2\exp\left(\mu\frac{1+z}{z-1}\right)C(\bar z-1-\mu)\overline{\exp\left(\mu\frac{1+z}{z-1}\right)}\,\bar z\,d\theta\\
&= \frac{C^2}{2}\frac{1}{2\pi}\int_{\mathbb{T}}(z^2-2z+1)(\bar z-(1+\mu))\,\bar z\,d\theta\\
&= \frac{C^2}{2}(3+2\mu).
\end{aligned}$$

Since $\int_{\mathbb{D}}|f^*(z)|^2\,dA = 1$, we get that $C = \sqrt{\frac{2}{3+2\mu}}$, where $\mu > 0$. Substituting C into (5.2), we obtain

$$(f^*)'(0) = \sqrt{\frac{2}{3+2\mu}}\,e^{-\mu}(1+2\mu+2\mu^2). \tag{5.3}$$

It is not hard to see that this function of μ, when $\mu > 0$, is maximized when $\mu = 1$. We thus obtain

$$\max\{\mathrm{Re} f'(0) : \|f\|_{A^2}\le 1,\ f \text{ non-vanishing in } \mathbb{D}\} = \sqrt{2}\,\frac{\sqrt{5}}{e}. \tag{5.4}$$

It is also natural then to expect that an extremal function for any m in the problem

$$\max\{\mathrm{Re} f^{(m)}(0) : \|f\|_{A^2}\le 1,\ f \text{ non-vanishing in } \mathbb{D}\} \tag{5.5}$$

would be $c_m f^*(z^m)$, where f^* is the extremal for (5.4)and c_m is the normalizing constant. From this, we leap to the following rather bold conjecture, although present evidence in its favor is not abundant.

Conjecture 5.2.

$$\limsup_{m\to\infty}\frac{\max\left\{\mathrm{Re}\, f^{(m)}(0)/m! : \|f\|_{A^2}\le 1,\ f \text{ non-vanishing in } \mathbb{D}\right\}}{\sqrt{m}} \le \frac{\sqrt{5}}{e}.$$

Since we still have not been able to completely solve the above problem for $m = 1$,

Conjecture 5.3.

$$\limsup_{m\to\infty} \frac{¤_m}{(\frac{mp+2}{2})^{\frac{1}{p}}} < 1.$$

Denote by $¸_m$ the analog of $¤_m$ in the H^p_0-context. A priori, of course, $¤_m \geq ¸_m$.

Question. What are the asymptotics of $¤_m$? Is $¤_m \sim ¸_m m^{1=p}$?

We think that perhaps with the advances in the theory of Bergman spaces in the last decade, the time has come for a thorough study of these fundamental extremal problems.

References

1. D. Aharonov, C. Bénéteau, D. Khavinson, and H.S. Shapiro, Extremal problems for nonvanishing functions in Bergman spaces, Selected Topics in Complex Analysis, 59–86, *Oper. Theory Adv. Appl.*, **158**, Birkhäuser, Basel, 2005.
2. D. Aharonov, H.S. Shapiro, and A. Solynin, A minimal area problem in conformal mapping, J. Analyse Math. **78** (1999), 157–176.
3. D. Aharonov, H. S. Shapiro, and A. Solynin, A minimal area problem in conformal mapping II, J. Analyse Math. **83** (2001), 259–288.
4. R. Barnard, C. Richardson, and A. Solynin, A minimal area problem for nonvanishing functions, Algebra i Analiz **18** (2006), no. 1, 34–54.
5. C. Bénéteau and B. Korenblum, Some coefficient estimates for H^p functions, Complex analysis and dynamical systems, 5–14, *Contemp. Math.*, **364**, American Mathematical Society, Providence, R.I., 2004.
6. J. E. Brown, On a coefficient problem for nonvanishing H^p functions, Complex Variables Theory Appl. **4** (1985), no. 3, 253–265.
7. P. Duren, *Theory of H^p Spaces*, Academic Press, New York-London 1970; Second Edition, Dover Publications, Mineola, N.Y., 2000.
8. P. Duren and A. Schuster, *Bergman Spaces*, American Mathematical Society, Providence, R.I., 2004.
9. H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, Springer-Verlag, New York, 2000.
10. Ch. Horowitz, Coefficients of nonvanishing functions in H^∞, Israel J. Math. **30** (1978), no. 3, 285–291.
11. J. A. Hummel, S. Scheinberg, and L. Zalcman, A coefficient problem for bounded nonvanishing functions, J. Analyse Math. **34** (1977), 169–190. 3T 72.36 -3771 -11.95 TD[(

19. K. Yu. Osipenko, M. I. Stessin, Recovery problems in Hardy and Bergman spaces, Mat. Zametki **49** (1991), no. 4, 95{104 (in Russian); translation in Math. Notes **49** (1991), no. 3-4, 395{401.
20. R. Peretz, The Krzyz̲ problem and polynomials with zeros on the unit circle, Complex Variables Theory Appl. **47** (2002), no. 3, 271{276.
21. R. Peretz, Applications of subordination theory to the class of bounded nonvanishing functions, Complex Variables Theory Appl. **17** (1992), no. 3-4, 213{222.
22. R. Peretz, Some properties of extremal functions for Krzyz̲ problem, Complex Variables Theory Appl. **16** (1991), no. 1, 1{7.
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