JENSEN TYPE INEQUALITIES AND RADIAL NULL SETS

Catherine Bénéteau and Boris Korenblum

Abstract. We extend Jensen's formula to obtain an upper estimate of $\log|f(0)|$ for analytic functions in the unit disk $\mathbb{D}$ that are subject to a growth restriction. Suppose we have a closed subset E of the unit circle and f in addition is continuous in the union of the open disk and E. We obtain a formula that gives an upper estimate of of $\log|f(0)|$ in terms of the values of f on E and the so-called k-entropy of E. When the set E is taken to be the whole unit circle, we get the classical Jensen's inequality. Our formula is then applied to the study of radial null sets. 2000 Mathematics Subject Classification: 30H05, 30E25, 46E15.

1 Growth Spaces

In what follows, k denotes an increasing twice differentiable function that maps $[0, 1)$ [illegible]

2 Two Problems

(A) Find good (upper and lower) estimates for the quantity

$$J(Z;k) = \sup\{\log|f(0)| : f \in UBA^{<k>}; f|_Z = 0\}$$

where $Z = \{a_n\} \subset \mathbb{D}$ is a given sequence.
(B) Find good estimates for

$$J(E;\varphi;k) = \sup\{\log|f(0)| : f \in UBA^{<k>} \cap C(\mathbb{D} \cup E); |f|\,|_E = \varphi\}$$

where $E \subset \partial\mathbb{D}$ is a closed set and φ is a non-negative continuous function on $E.$
Note that for $k \equiv 0;$ $(A^{<0>} = H^\infty)$ both problems have exact solutions:

$$J(Z;0) = -\sum_n \log\frac{1}{|a_n|}$$

$$J(E;\varphi;0) = \int_E \log\varphi(\zeta)\,dm(\zeta)$$

where dm is the normalized Lebesgue measure on $\partial\mathbb{D}.$ (Here, we assume k and the radial projection of S :

$$\Pr S = \{\frac{z}{|z|} : z \in S\}.$$

Then we have

$$J(Z;\lambda_\alpha) \le \inf_{S\subset Z} \{\alpha[\hat{\ }(\Pr S) + \log \hat{\ }(\Pr S)] - T(s) + \alpha \log^+ T(s)\} + C_\alpha$$

and

$$J(Z;\lambda_\alpha) \ge \inf_{S\subset Z} \{\alpha[\hat{\ }(\Pr S) - \log \hat{\ }(\Pr S)] - T(s)\} - C_\alpha$$

where $C_\alpha > 0$ depends only on α; and the infima are taken over all finite subsets S of Z.

<u>COROLLARY</u> 3.1 *For a sequence Z such that* 0 *is not in Z; define*

$$D^+(Z) = \inf\{m : \inf_{S\subset Z}(m\hat{\ }(\Pr S) - T(s)) > -\infty\}.$$

Then $D^+(Z) \le \alpha$ is necessary and $D^+(Z) < \alpha$ is sufficient for Z to be an $A^{-\alpha}$ zero set.

Note that for other spaces $A^{<k>}$ such that k has faster than logarithmic growth, a similar description of zero sets is not known.

4 Problem (B) for $A^{<k>}$

<u>THEOREM</u> 4.1

$$J(E;\varphi;k) \le \int_E \max\{\log \varphi(\zeta); \log p\} dm(\zeta) - (\log p)\frac{\alpha}{1-\alpha}(1-|E|) + (\frac{L}{\alpha})^{\log_2 C} Entr_k(E)$$

where $0 < p \le 1$, $0 < \alpha \le \frac{1}{2}$ *are arbitrary, C is the constant in (3), L is an absolute constant, and $Entr_k(E)$ is the k-entropy of E, defined as follows:*

$$Entr_k(E) = \sum_n \int_{1-|I_n|}^{1} k(t)dt$$

where $\{I_n\}$ are the complementary arcs of E.

Special cases: (1) $E = \partial\mathbb{D}.$ Letting $p \to 0^+,$ we get

$$J(\partial\mathbb{D};\varphi;k) \le \int_{\partial\mathbb{D}} \log\varphi(\zeta)dm(\zeta)$$

which is the classical Jensen's inequality (in fact, equality.)
(2) If $0 \le \varphi(\zeta) \le 1$ on E and $p = \max_{\zeta\in E}\varphi(\zeta),$ we obtain

$$J(E;\varphi;k) \le (\log p)\frac{|E| - \alpha}{1-\alpha} + \left(\frac{L}{\alpha}\right)^{\log_2 C} Entr_k(E).$$

Choosing $\alpha = |E|/2,$ we get

$$J(E;\varphi;k) \le \frac{1}{2}(\log p)|E| + \left(\frac{2L}{|E|}\right)^{\log_2 C} Entr_k(E).$$

(3) If $p = 1$ and $\alpha = \frac{1}{2},$ then

$$J(E;\varphi;k) \le \int_E \log^+\varphi(\zeta)dm(\zeta) + (2L)^{\log_2 C} Entr_k(E).$$

Proof: Write

$$\partial\mathbb{D} - E = \bigcup_n I_n$$

where the I_n are open disjoint arcs on the unit circle. Call a_n and b_n the endpoints of $I_n.$ Let $0 < \alpha \le \frac{1}{2}.$ Let γ_n be the open arc of the circle inside the unit disk passing through a_n and b_n and forming an angle of $\pi\alpha$ (we will think of it as the normalized angle α) with the arc $I_n.$ Let $\Gamma = \bigcup_n \gamma_n.$ $\Gamma \cup E$ forms the boundary of an open subset Ω of the unit disk containing the origin. For the proof, we construct three functions $U_1, U_2,$ and V as follows.
Step 1: Construction of U_1 and U_2.
Define

$$U_1(z) = \int_E Re\left(\frac{\zeta+z}{\zeta-z}\right)dm(\zeta).$$

U_1 is the harmonic measure of E with respect to $\mathbb{D}.$

<u>LEMMA</u> 4.1

$$\lim_{r\to 1^-} U_1(r\zeta) = \chi_E(\zeta) \text{ a.e. on } \partial\mathbb{D}$$

where χ_E is the characteristic function of $E.$ In addition, $U_1(z) \le \alpha$ for $z \in \Gamma.$

Assume $Entr_k(E)$ is finite and define the following harmonic function

$$V(z) = \int_{\partial \mathbb{D}}$$

By relabeling L; we get the statement of the lemma. $\Box$

Step 3: Construction of H and application of the maximum principle. Finally, let us define

$$H(z) = U_2(z) - (\log p)\frac{\alpha}{1-\alpha}(1 - U_1(z)) + ({}^{L}$$

$\lim_{r \to 1-} f(r\zeta$